

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE

No. 1572

STABILITY DERIVATIVES OF TRIANGULAR WINGS AT SUPERSONIC SPEEDS

By Herbert S. Ribner and Frank S. Malvestuto, Jr.

Langley Memorial Aeronautical Laboratory
Langley Field, Va.



Washington

May 1948

FOR REFERENCE

NOT TO BE TAKEN FROM THIS ROOM

NACA TN 1572

STABILITY DERIVATIVES OF TRIANGULAR WINGS AT
SUPERSONIC SPEEDS

By Herbert S. Ribner and Frank S. Malvestuto, Jr.

May 1948

Page 7: The important derivatives $C_{L_{\dot{\alpha}}}$ and $C_{m_{\dot{\alpha}}}$ are obtained incorrectly. The corrected values for these derivatives should read as follows:

$$C_{L_{\dot{\alpha}}} = -\frac{\pi A}{2} \frac{E'''(BC) - M^2 H(BC)}{M^2 - 1} \quad (4a)$$

$$C_{m_{\dot{\alpha}}} = \frac{\pi A}{16} \frac{E'''(BC) - M^2 H(BC)}{M^2 - 1} \quad (5a)$$

The same correction should be made for these derivatives in table I in the column "Principal body axes." In the column "Stability axes" of table I the corrected expressions are

$$C_{L_{\dot{\alpha}}} = -\frac{\pi A}{2} \frac{E'''(BC) - M^2 H(BC)}{M^2 - 1}$$

$$C_{m_{\dot{\alpha}}} = \frac{\pi A}{16} \left(1 + 8 \frac{x_{cg}}{\bar{c}} \right) \frac{E'''(BC) - M^2 H(BC)}{M^2 - 1}$$

The error in the derivation consists in the assumption, carried over from reference 1, that the surface potential, equation (3), is not altered by a small normal acceleration. This assumption is true for the narrow triangles treated in reference 1 but fails for the general triangles treated in the present paper. The assumption is

equivalent to the neglect of the time dependency terms $\frac{2V}{\bar{a}^2} \phi_{xt}$

and $\frac{1}{\bar{a}^2} \phi_{tt}$ (where t represents time and \bar{a} is the speed of sound) in the linearized partial differential equation for unsteady motions:

$$B^2 \phi_{xx} - \phi_{yy} - \phi_{zz} + \frac{2V}{a^2} \phi_{xt} + \frac{1}{a^2} \phi_{tt} = 0 \quad (a)$$

which leaves only the steady-state (Prandtl-Glauert) equation

$$B^2 \phi_{xx} - \phi_{yy} - \phi_{zz} = 0 \quad (b)$$

The correct potential to replace equation (3) must satisfy both equation (a) and the boundary condition on the wing

$$\left(\frac{\partial \phi}{\partial z} \right)_{z=0} = -\dot{\alpha} V t \quad (c)$$

In an unpublished paper, Mr. Clifford S. Gardner has, in effect, shown that a suitable solution is

$$\frac{\phi}{\dot{\alpha}} = \frac{M^2}{B^2} \psi + \left(t - \frac{M^2 x}{VB^2} \right) \chi \quad (d)$$

where ψ is the steady-state potential corresponding to a unit pitching velocity about the y-axis and χ is the steady-state potential corresponding to unit angle of attack. Both ψ and χ satisfy equations of the form (b). That equation (d) is a solution can be verified by direct substitution into equations (a) and (c). Thus, Gardner has shown that the time-dependent potential for an angle of attack $\dot{\alpha} t$ may be compounded of two time-free, or steady-state, potentials, one for a constant angle of attack and the other for steady pitching.

The lift distribution at time $t = 0$ for the angle of attack $\dot{\alpha} t$ is obtained from the potential by

$$\begin{aligned} \Delta P &= 2\rho(V\phi_x + \phi_t) \\ &= 2\rho V \dot{\alpha} \left(\frac{M^2}{B^2} \psi_x - \frac{M^2 x}{VB^2} \chi_x - \frac{\chi}{VB^2} \right) \\ &= \frac{\dot{\alpha}}{B^2} \left[M^2 (\Delta P)_{q=1} - \frac{M^2 x}{V} (\Delta P)_{\alpha=1} - \Delta P^* \right] \quad (e) \end{aligned}$$

where

$(\Delta P)_{q=1}$ lift distribution for unit pitching velocity

$(\Delta P)_{\alpha=1}$ lift distribution for unit angle of attack

ΔP^* value of ΔP per unit $\dot{\alpha}$ used in deriving the incorrect equations (4) and (5)

Integration of equation (c) over the plan form to obtain the lift and moment and reduction to coefficient form yields

$$C_{L\dot{\alpha}} = \frac{M^2}{B^2} C_{Lq} + 2 \frac{M^2}{B^2} C_{m\alpha} - \frac{1}{B^2} (C_{L\dot{\alpha}})^* \quad (f)$$

$$C_{m\dot{\alpha}} = \frac{M^2}{B^2} C_{mq} + \frac{2M^2}{B^2 Sc^2} \iint_{\substack{\text{plan} \\ \text{form}}} x^2 \left(\frac{\Delta P}{\frac{1}{2} \rho V^2} \right)_{\alpha=1} dx dy - \frac{1}{B^2} (C_{m\dot{\alpha}})^* \quad (g)$$

where the * designates the incorrect values in equations (4) and (5) respectively, and the y-axis is taken through the center of gravity. Values of C_{Lq} , $C_{m\alpha}$, and C_{mq} , may be obtained from table I and $(C_{L\dot{\alpha}})^*$ and $(C_{m\dot{\alpha}})^*$, from equations (4) and (5). The

quantity $\left(\frac{\Delta P}{\frac{1}{2} \rho V^2} \right)_{\alpha=1}$ is obtained by setting $\alpha = 1$ and $a = c \left(\frac{2}{3} c + x \right)$

in equation (1). Substitution and integration then yields the corrected values for $C_{L\dot{\alpha}}$ and $C_{m\dot{\alpha}}$ (equations (4a) and (5a)) as set forth at the beginning of this erratum.

ERRATA

NACA TN No. 1572

STABILITY DERIVATIVES OF TRIANGULAR WINGS AT SUPERSONIC SPEEDS

By Herbert S. Ribner and Frank S. Malvestuto, Jr.

May 1948

Page 14: The section entitled "DERIVATIVES C_{Y_β} , C_{n_β} , C_{Y_r} , and C_{n_r} " and the part of table I in which these derivatives are presented have been found to be in error. The null result deduced for these derivatives (in the neglect of friction drag) is incorrect as a result of the omission of a compressibility factor in the treatment of the leading-edge suction force. Thus, on page 14 (last sentence of first paragraph), the remark that the initially symmetric distribution of leading-edge suction persists in sideslip (for small values of sideslip) is a misstatement. The initially symmetric distribution of leading-edge velocity persists in sideslip, but compressibility upsets the symmetry of the leading-edge suction. According to equation (17) the suction per unit length of edge is

$$f = \pi \rho K^2 \sqrt{1 - M'^2}$$

For infinitesimal sideslip the constant K , related to the edge velocity, is unchanged, but M' , the component Mach number perpendicular to the edge, is altered: M' increases on the right edge and decreases on the left edge. The error under discussion consists in the neglect of the change in M' .

Because of the change of M' with sideslip angle β the edge suction may be written

$$f = f_{\beta=0} + \beta \left(\frac{\partial f}{\partial \beta} \right)_{\beta=0} \quad (21)$$

By differentiation of equation (17), with K constant and $M' = M \sin(\epsilon \pm \beta)$, there results

$$f = f_{\beta=0} \mp \beta f_{\beta=0} \left(\frac{M'^2 \cot \epsilon}{1 - M'^2} \right)_{\beta=0} \quad (22)$$

where the upper part of the double sign refers to the right edge and the lower part to the left edge.

The quantity $f_{\beta=0}$ is obtained by setting $p = 0$ in the last equation of page 12:

$$f_{\beta=0} = \frac{\pi \rho C x \alpha^2 V^2 \sqrt{1 - B^2 C^2}}{2 [E'(BC)]^2} \quad (23)$$

Substitution of equation (23) in the last term of equation (22) and simplification, with $\tan \epsilon = C$, yields

$$f = f_{\beta=0} \mp \beta \frac{\pi \rho V^2 \alpha^2 x C^2 M^2}{2 [E'(BC)]^2 \sqrt{1 - B^2 C^2}} \quad (24)$$

Equation (24) gives the suction per unit length of edge for a triangular wing with an angle of sideslip β .

For the case of a small angular velocity of yawing r the edge suction may be approximated by

$$f = f_{r=0} + r \left(\frac{\partial f}{\partial r} \right)_{r=0}$$

where $f_{r=0}$ is the same as $f_{\beta=0}$ and is given by equation (23). If the center of rotation is $\left(\frac{2}{3}c, 0, 0\right)$, the component Mach number normal to the edges is

$$M' = M \left[\sin \epsilon \pm \frac{r}{V} \left(\frac{2}{3}c \cos \epsilon - x \sec \epsilon \right) \right]$$

This value of M' is to be incorporated in equation (17) for f before carrying out the indicated differentiation $\frac{\partial f}{\partial r}$. The final result is

$$f = f_{\beta=0} \mp r \frac{\left(\frac{2}{3}c - x \sec^2 \epsilon\right) \pi \rho V \alpha^2 x C^2 M^2}{2[E'(BC)]^2 \sqrt{1 - B^2 C^2}} \quad (25)$$

The suction equations (24) and (25) have been used in procedures similar to those presented on page 13 to yield corrected values of the stability derivatives $C_{Y\beta}$, $C_{n\beta}$, C_{Yr} , and a correction term to be added to the right-hand side of equation (20). The results are incorporated in a correction to table I in the column labeled "Principal body axes". Since one or another of these four derivatives figures in their transformation to stability axes, several of the equations in the column of table I labeled "Stability axes" have also required correction.

The yawing derivatives C_{Yr} and C_{nr} as corrected herein are still subject to the uncertainty inherent in the limitations of the linearized theory for yawing motion pointed out on pages 9 and 10.

Page 18: Formulas for $C_{n\beta}$, C_{np} , C_{nr} , $C_{Y\beta}$, C_{Yp} , and C_{Yr} should be changed to read as shown in attached correction to table I.

Corrected version of part of table I following the derivative C_{L_r} .

Derivative	Source (if not derived herein)	Principal body axes (origin at $\frac{2}{3}c$)	Stability Axes (origin at distance x_{cg} ahead $\frac{2}{3}c$ point)
$C_{n\beta}$		$\frac{\pi}{48} \alpha^2 A^2 M^2 Q(BC)$	$\frac{\pi \alpha^2}{3} \left[E''(BC) + \left(\frac{A^2}{16} + \frac{x_{cg}}{c} \right) M^2 Q(BC) \right]$
C_{np}		$-\pi \alpha \left(\frac{1}{9A} + \frac{A}{16} \right) J(BC)$	$-\pi \alpha \left[\left(\frac{1}{9A} + \frac{A}{16} + \frac{8}{9A} \frac{x_{cg}}{c} \right) J(BC) - \frac{A}{32} I(BC) \right] - \alpha C_{D0} \left(\frac{1}{6} + \frac{4}{9A^2} \right)$
C_{nr}	Term in C_{D0} derived in reference 1.	$-C_{D0} \left(\frac{1}{6} + \frac{4}{9A^2} \right) - \frac{\pi \alpha^2 M^2}{9} \left(\frac{1}{A} + \frac{A}{8} + \frac{9A^3}{256} \right) Q(BC)$	$-C_{D0} \left(\frac{1}{6} + \frac{4}{9A^2} \right) - \pi \alpha^2 \left(\frac{1}{9A} + \frac{A}{16} + \frac{8}{9A} \frac{x_{cg}}{c} \right) [E''(BC) - J(BC)] - \pi \alpha^2 \frac{A}{32} I(BC) - \frac{\pi \alpha^2 M^2}{9} \left(\frac{1}{A} + \frac{A}{8} + \frac{9A^3}{256} + A \frac{x_{cg}}{c} + \frac{8}{A} \frac{x_{cg}^2}{c^2} \right) Q(BC)$
$C_{Y\beta}$		$-\frac{\pi}{4} \alpha^2 A M^2 Q(BC)$	$-\frac{\pi}{4} \alpha^2 A M^2 Q(BC)$
C_{Yp}		$\frac{2\pi \alpha}{3} J(BC)$	$\frac{2\pi \alpha}{3} J(BC)$
C_{Yr}		$\frac{\pi}{24} \alpha^2 A^2 M^2 Q(BC)$	$\frac{2\pi \alpha^2}{3} \left[-J(BC) + \left(\frac{A^2}{16} + \frac{x_{cg}}{c} \right) M^2 Q(BC) \right]$

where $Q(BC) \equiv \frac{[E''(BC)]^2}{\sqrt{1 - B^2 C^2}}$ is a new function.

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE NO. 1572

STABILITY DERIVATIVES OF TRIANGULAR WINGS AT SUPERSONIC SPEEDS

By Herbert S. Ribner and Frank S. Malvestuto, Jr.

S U M M A R Y

The analysis of the stability derivatives of low-aspect-ratio triangular wings at subsonic and supersonic speeds, given in NACA TN No. 1423, is extended to apply to triangular wings having large vertex angles and traveling at supersonic speeds. The lift, rolling moment due to sideslip, and damping in roll and pitch for this more general case have been treated elsewhere on the basis of the theory of small disturbances. The surface potentials for angle of attack and rolling taken therefrom are used to obtain the lift due to downward acceleration, the several side-force and yawing-moment derivatives that depend on leading-edge suction, and a tentative value for the rolling moment due to yawing. All the known stability derivatives of the triangular wing at supersonic speeds, regardless of source, are summarized for convenience and presented with respect to both body axes and stability axes. The results are limited to Mach numbers for which the triangular wing is contained within the Mach cone from its vertex. The spanwise variation of Mach number in the case of yawing is neglected, although the effect must be of importance.

I N T R O D U C T I O N

An earlier investigation (reference 1) has provided theoretical stability derivatives of low-aspect-ratio wings of triangular plan form at subsonic and supersonic speeds. The restriction to low aspect ratio was a consequence of the limitations of the theory. Several investigators have since obtained pressure distributions for angle of attack, rolling, pitching, and sideslip at supersonic speeds (references 2 to 6 and unpublished analyses), without restriction to low aspect ratio. These investigations have employed variants of the linear theory of supersonic flow and have, in fact, constituted important steps in the development of the theory.

If the rotations are taken about the vertex, the pressure distribution for each motion in the more general case is found to have the same shape as the corresponding low-aspect-ratio approximation, so long as the triangular wing is contained within the Mach cone from the vertex.

The magnitudes differ by factors which are functions solely of the ratio of the tangent of the semivertex angle of the triangle to the tangent of the Mach angle. The same similarity exists between the distributions of surface potential. It is thus relatively simple to extend most of the derivations of reference 1 to remove the restriction to low aspect ratio for supersonic speeds. Such an extension is made in the present paper.

The lift-curve slope, the damping in roll and pitch, and (in effect) the rolling moment due to sideslip have been evaluated in references 2 to 6, so that the principal contributions of the present paper are the several side-force and yawing-moment derivatives and a tentative value of the rolling moment due to yawing. All the known stability derivatives of the triangular wing at supersonic speeds, regardless of source, are collected herein for convenience and presented with respect to both body axes and stability axes. Wings with dihedral are not treated (although they were included in reference 1) and the results are limited to Mach numbers for which the wing is contained within the Mach cone from its vertex.

S Y M B O L S

x, y, z	rectangular coordinates (fig. 1)
u, v, w	incremental flight velocities along x -, y -, and z -axes, respectively, figure 2; induced flow velocities along x -, y -, and z -axes of figure 1, respectively
p, q, r	angular velocities about x -, y -, and z -axes, respectively, figure 2
V	flight speed
M	stream Mach number ($V/\text{Speed of sound}$)
M'	component Mach number normal to wing leading edge $\left(\frac{MC}{\sqrt{1 + C^2}} \right)$
B	cotangent of Mach angle $\left(\sqrt{M^2 - 1} \right)$
α	angle of attack (Flight w/V)
β	angle of sideslip (Flight v/V)
ϵ	semivertex angle of triangle
μ	Mach angle $\left(\cot^{-1} \sqrt{M^2 - 1} \right)$

- ΔP local pressure difference between lower and upper surfaces of airfoil, positive in sense of a lift
- ρ density of air
- a semiwidth of triangle at distance x from vertex
- b span (base of triangle)
- c root chord (height of triangle)
- \bar{c} mean aerodynamic chord $\left(\bar{c} \equiv \frac{2}{S} \int_0^{b/2} (\text{Local chord})^2 dy = \frac{2}{3} c \right)$
- C edge slope $\left(\frac{a}{x} = \frac{da}{dx} = \frac{A}{4} = \frac{b}{2c} \right)$
- A aspect ratio $\left(\frac{2b}{c} \right)$
- S area of triangle $\left(\frac{1}{2} bc \right)$
- ϕ velocity potential
- $\eta = \cos^{-1} \frac{x}{a}$
- $k = \sqrt{1 - B^2 C^2}$
- $E^*(BC)$ complete elliptic integral of the second kind with modulus k ; $\left(\int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 z} dz \right)$
- $F^*(BC)$ complete elliptic integral of the first kind with modulus k ; $\left(\int_0^{\frac{\pi}{2}} \frac{dz}{\sqrt{1 - k^2 \sin^2 z}} \right)$
- $E''(BC) = \frac{1}{E^*(BC)}$
- $G(BC) = \frac{1 - B^2 C^2}{(1 - 2B^2 C^2)E^*(BC) + B^2 C^2 F^*(BC)}$
- $H(BC) = 3G(BC) - 2E''(BC)$

$$I(BC) = \frac{2(1 - B^2C^2)}{(2 - B^2C^2)E^*(BC) - B^2C^2F^*(BC)}$$

$$J(BC) = E^*(BC)I(BC)\sqrt{1 - B^2C^2}$$

K constant defined in equation (16)

N yawing moment

Y lateral force

f suction force per unit length of edge

C_L lift coefficient $\left(\frac{\text{Lift}}{\frac{1}{2} \rho V^2 S} \right)$

C_m pitching-moment coefficient $\left(\frac{\text{Pitching moment}}{\frac{1}{2} \rho V^2 S \bar{c}} \right)$

C_l rolling-moment coefficient $\left(\frac{\text{Rolling moment}}{\frac{1}{2} \rho V^2 S b} \right)$

C_n yawing-moment coefficient $\left(\frac{N}{\frac{1}{2} \rho V^2 S b} \right)$

C_Y lateral-force coefficient $\left(\frac{Y}{\frac{1}{2} \rho V^2 S} \right)$

C_{D0} profile-drag coefficient $\left(\frac{\text{Profile drag}}{\frac{1}{2} \rho V^2 S} \right)$

v_N induced surface velocity normal to wing leading edge

s perpendicular distance of point (x,y) from wing leading edge

x_{cg} distance of center of gravity forward of $\frac{2}{3}c$

Subscripts:

R right edge

L left edge

Whenever α , $\dot{\alpha}$, q , p , β , and r are used as subscripts, a nondimensional derivative is indicated and this derivative is the slope through zero. For example,

$$C_{m\dot{\alpha}} = \left[\frac{\partial C_m}{\partial \left(\frac{\dot{\alpha} c}{2V} \right)} \right]_{\dot{\alpha} \rightarrow 0} \quad C_{mq} = \left[\frac{\partial C_m}{\partial \left(\frac{q c}{2V} \right)} \right]_{q \rightarrow 0} \quad C_{lp} = \left[\frac{\partial C_l}{\partial \left(\frac{pb}{2V} \right)} \right]_{p \rightarrow 0}$$

$$C_{l\beta} = \left[\frac{\partial C_l}{\partial \beta} \right]_{\beta \rightarrow 0} \quad C_{lr} = \left[\frac{\partial C_l}{\partial \left(\frac{rb}{2V} \right)} \right]_{r \rightarrow 0}$$

A dot above a symbol denotes differentiation with respect to time.

All angles are measured in radians.

A N A L Y S I S

SCOPE

The stability derivatives of triangular wings at supersonic speed that have been treated theoretically herein or elsewhere are listed in table I, together with the expressions that have been found for them. All the derivations make use of body axes. The derivations that follow give the values with reference to the principal body axes of figure 2 with origin at the aerodynamic center $\left(\frac{2}{3}c, 0, 0 \right)$. Conversion has been made to the system of stability axes shown in figure 3 with origin a distance x_{cg} ahead of the $\frac{2}{3}c$ point. Table I comprises parallel columns which present formulas relative to both systems. The expressions are limited to Mach numbers for which the triangle is contained within the Mach cone from its vertex.

DERIVATIVES $C_{L\alpha}$, $C_{L\dot{\alpha}}$, and $C_{m\dot{\alpha}}$

The pressure distribution on a thin delta wing at an angle of attack in a supersonic stream has been obtained in references 2 to 4 by the

linearized theory without restriction on the vertex angle of the triangle. The approximation originally given for the slender (low-aspect-ratio) triangle (reference 7) and used as the basis for reference 1 is found to apply to the general case upon division by a constant (an elliptic integral) that depends on the ratio of the semivertex angle to the Mach angle. That is,

$$\frac{\Delta P}{\frac{1}{2} \rho V^2} = \frac{4aC\alpha}{E'(BC) \sqrt{a^2 - y^2}} \quad (1)$$

where $E'(BC)$ is the complete elliptic integral of the second kind with modulus

$$\begin{aligned} k &= \sqrt{1 - B^2 C^2} \\ &= \sqrt{1 - (M^2 - 1) C^2} \end{aligned}$$

Thus the lift-curve slope for the more general case is the value given by references 7 and 1 divided by $E'(BC)$:

$$\begin{aligned} C_{L\alpha} &= \frac{\pi A}{2E'(BC)} \\ &= \frac{\pi}{2} AE''(BC) \end{aligned} \quad (2)$$

The surface potential given in equation (3) of reference 1 is likewise extended to include nonslender triangles at supersonic speeds upon division by $E'(BC)$. The revised potential is

$$\begin{aligned} (\phi)_{\pm z=0} &= \pm \frac{V\alpha a \sin \eta}{E'(BC)} \\ &= \pm \frac{V\alpha \sqrt{a^2 - y^2}}{E'(BC)} \end{aligned} \quad (3)$$

The elliptic integral $E'(BC)$ depends only on the parameter $BC = \frac{\tan \epsilon}{\tan \mu}$ (ratio of the tangent of the semivertex angle of the delta wing to the

tangent of the Mach angle) and is therefore a constant for a given wing at a given speed. The derivations in reference 1 for C_{L_α} and C_{m_α} , which are based on the potential ϕ , thus merely acquire a factor

$$E''(BC) = \frac{1}{E^*(BC)} :$$

$$C_{L_\alpha} = \frac{\pi A}{2} E''(BC) \quad (4)$$

$$C_{m_\alpha} = -\frac{\pi A}{16} E''(BC) \quad (5)$$

DERIVATIVES C_{m_q} , C_{L_q} , and C_{l_p}

The derivatives C_{m_q} , C_{L_q} , and C_{l_p} are derived in reference 5. With respect to the axes of figure 2

$$C_{m_q} = -\frac{3\pi A}{16} G(BC) \quad (6)$$

$$C_{L_q} = \frac{\pi A}{2} H(BC) \quad (7)$$

$$C_{l_p} = -\frac{\pi A}{32} I(BC) \quad (8)$$

where

$$G(BC) = \frac{1 - B^2 C^2}{(1 - 2B^2 C^2)E^*(BC) + B^2 C^2 F^*(BC)} \quad (9)$$

$$H(BC) = 3G(BC) - 2E''(BC) \quad (10)$$

$$I(BC) = \frac{2(1 - B^2 C^2)}{(2 - B^2 C^2)E^*(BC) - B^2 C^2 F^*(BC)} \quad (11)$$

and $F'(BC)$ and $E'(BC)$ are the complete elliptic integrals of the first and second kinds, respectively, with modulus $k = \sqrt{1 - B^2 C^2}$.

DERIVATIVE $C_{l\beta}$

The pressure distribution over a thin delta wing in yaw (sideslip) at an angle of attack at supersonic speed has been obtained in reference 6 and unpublished work. If the angle of yaw is assumed to be small ($\beta \ll \frac{1}{M}$), the rolling-moment coefficient can be expressed in the approximate form

$$C_l \approx -\frac{\pi\alpha\beta}{3} E''(BC)$$

Thus, the derivative with respect to β is

$$C_{l\beta} = -\frac{\pi\alpha}{3} E''(BC) \quad (12)$$

An alternative derivation based on the surface potential, equation (3), for the unyawed wing will be given because the method provides the starting point for a derivation of C_{l_r} , C_{n_r} , and C_{y_r} .

The potential for the disturbance velocity may be expressed relative to axes aligned with the stream (wind axes) or with respect to axes that yaw with the body (body axes). For small angles of yaw ($\beta \ll \frac{1}{M}$), the linearized equation for the potential has the same form relative to either system of axes. The potential is determined by the normal velocity of points of the surface and by the orientation of the surface; for negligible thickness, this normal velocity is just αV for all angles of yaw. The potential expressed relative to wind axes thus varies as the wing yaws relative to these axes. The potential expressed relative to body axes is constant for small yaw because the orientation of the wing relative to the axes does not change.

For wind axes, Bernoulli's law has the form

$$\Delta P = 2\rho V \frac{\partial \phi}{\partial x}$$

and the change in the pressure distribution with yaw results from the change in the potential function with yaw. For body axes with small yaw, Bernoulli's law has the approximate form

$$\Delta P = 2\rho V \left(\frac{\partial \phi}{\partial x} - \beta \frac{\partial \phi}{\partial y} \right) \quad (13)$$

and the change in pressure distribution with yaw results from the term $-\beta \frac{\partial \phi}{\partial y}$ since ϕ does not change.

In reference 1 in the section entitled "DERIVATIVE C_{l_β} ," the derivation employs body axes and equation (13) of the present paper. The surface potential used (equation (3) of reference 1) is the approximation for narrow vertex angle. Equation (3) herein for a general vertex angle may be used instead. Equation (3) herein differs only in the factor $1/E^*(BC)$, whence the earlier expression for C_{l_β} (equation (19), reference 1, with $\Gamma = 0^\circ$) acquires this factor to agree with equation (12).

DERIVATIVE C_{l_r}

The foregoing discussion of the triangular wing in yaw (sideslip) may be extended to provide a preliminary treatment of the case of a small angular velocity of yaw r . The corresponding extension for narrow vertex angle is made in reference 1. The treatment is generalized to an arbitrary vertex angle for supersonic speeds, as before, by using equation (3) herein for the surface potential. Two changes then appear in the pressure equation, equation (20), of reference 1. The right-hand side is divided by $E^*(BC)$, and the term $aC = xC^2$ must be retained, since C^2 is no longer small compared with unity ($C = \tan$ of semivertex angle). With these changes, the derivation leads to

$$C_{l_r} = \pi \alpha \left(\frac{1}{9A} + \frac{A}{16} \right) E''(BC) \quad (14)$$

In the derivation of equation (14), the spanwise variation in local Mach number caused by yawing is not taken into account although the variation in forward speed is taken into account. The surface potential that is used, equation (3), satisfies the linearized equation for a flow of uniform Mach number. This potential is inadequate to describe the compressibility effects associated with a spanwise variation of Mach number.

Thus, consider a high-aspect-ratio rectangular wing with tips cut off along the Mach lines. In straight flight the Ackeret theory can be applied. The pressure difference is given by

$$\Delta P = \alpha \frac{2\rho M^2 (\text{Speed of sound})^2}{\sqrt{M^2 - 1}} \quad (15)$$

In yawing flight the forward velocity varies linearly along the span. If the rate of yaw is made sufficiently low, the variation from wing tip to wing tip can be made so small that the flow is still nearly two-dimensional at any point. Thus the Ackeret theory is still applicable if the local Mach number is used at each spanwise station.

The variation in pressure with local Mach number can be obtained from equation (15). As the Mach number is increased, the pressure decreases from infinity at $M = 1$ to a minimum at $M = 1.4$ and then increases again. Thus below Mach number 1.4 the faster moving sections of the yawing wing have the lesser lift. This result is contrary to subsonic behavior and to that which would be predicted if the spanwise variation of Mach number were neglected. Thus the spanwise variation of the compressibility effect causes a reversal of the sign of the rolling moment due to yawing for rectangular wings at Mach numbers between 1 and 1.4, and at $M = 1.4$ the moment is zero. (This result refers to yawing in a system of stability axes, fig. 3. For body axes, fig. 2, the effect is similar but the reversal extends to $M = \infty$.)

A yawing triangular wing may be expected likewise to show an effect of the spanwise variation in Mach number. If the triangle is contained within the Mach cone from its vertex (the only case considered in this paper), however, the effect should be very much less than for the rectangular wing. In particular, where the predicted effect for the rectangular wing is a reversal of the sign of the rolling moment, the effect for the triangular wing is expected to be merely a change in the magnitude. A reversal in sign is not expected until the edges of the triangle protrude from the Mach cone. This behavior is inferred from the fact that the analyses of references 2 to 7 show many subsonic characteristics for triangles within the Mach cone and a marked change in characteristics for triangles with side edges outside the Mach cone.

DERIVATIVES C_{Y_p} and C_{n_p}

Extensive changes are necessary to generalize the treatment of C_{Y_p} and C_{n_p} in reference 1 to arbitrary vertex angles for supersonic speeds; therefore, the revised derivation is given in detail.

The derivatives C_{Y_p} and C_{n_p} relative to body axes for a very thin delta wing without dihedral arise entirely from suction on the wing

side edges. Consider a condition for which the induced velocity normal to the edge is of the form

$$v_N = \pm \frac{K}{\sqrt{s}} \quad (16)$$

in the immediate neighborhood of the edge, where s is the perpendicular distance from the edge and K is a constant. Reference 3 points out that for such a flow there is a suction force per unit length of edge,

$$f = \pi \rho K^2 \sqrt{1 - M^2} \quad (17)$$

so long as the delta wing does not protrude from the Mach cone from its vertex. In equation (17), M^2 is the Mach number of the component of the stream flow normal to the leading edge. The radical $\sqrt{1 - M^2}$ is the Prandtl-Glauert compressibility factor for the normal component of flow. Equation (17) is limited to real values of the radical by the condition expressed for the Mach cone.

For the delta wing in rolling motion the induced velocity component u has been obtained in reference 5 as

$$u_1 = \pm \frac{\pi \gamma C^2}{2 \sqrt{C^2 - \left(\frac{y}{x}\right)^2}} I(BC)$$

Angle of attack gives the additional contribution (reference 2)

$$u_2 = \pm \frac{\alpha V C^2}{E^*(BC) \sqrt{C^2 - \left(\frac{y}{x}\right)^2}}$$

The total induced velocity on the upper surface is thus the sum of u_1 and u_2 with the plus sign

$$u = \frac{C^2}{\sqrt{C^2 - \left(\frac{y}{x}\right)^2}} \left[\frac{\alpha V}{E^*(BC)} + \frac{\pi \gamma}{2} I(BC) \right]$$

Very near the side edge this velocity is approximately

$$u = \frac{c^{3/2}}{\sqrt{2(c - |\frac{y}{x}|)}} \left[\frac{\alpha V}{E^*(BC)} \pm \frac{pCx}{2} I(BC) \right]$$

where the plus sign refers to the right edge and the minus sign to the left edge.

If a similar calculation is made for $v = \frac{\partial \phi}{\partial y}$, it is found that as the side edge is approached the resultant induced velocity $\sqrt{u^2 + v^2}$ becomes normal to the edge. Thus the normal velocity near the edge is

$$v_N = \frac{\sqrt{1 + c^2}}{c} u$$

The perpendicular distance of point (x, y) from the side edge is

$$s = \frac{x(c - |\frac{y}{x}|)}{\sqrt{1 + c^2}}$$

The resultant induced velocity very near the edge may therefore be expressed approximately as

$$v_N = \pm \left[\frac{\alpha V}{E^*(BC)} \pm \frac{I(BC)pCx}{2} \right] (1 + c^2)^{1/4} \left(\frac{Cx}{2s} \right)^{1/2}$$

which is of the form of equation (16). The suction force per unit length of edge is from equation (17) thus

$$f = \frac{\pi}{2} \rho Cx \left\{ \frac{\alpha^2 V^2}{[E^*(BC)]^2} + \frac{[I(BC)]^2 p^2 c^2 x^2}{4} \pm \frac{I(BC) \alpha V p Cx}{E^*(BC)} \right\} \sqrt{(1 + c^2)(1 - M^2)}$$

where the plus sign refers to the right edge and the minus sign refers to the left edge. The factor $\sqrt{(1 + c^2)(1 - M^2)}$ can be reduced to $\sqrt{1 - B^2 C^2}$, where $B^2 = M^2 - 1$.

The lateral component of this suction force is given by

$$Y = \int_0^c (f_R - f_L) dx$$

$$= \frac{\pi}{3} \rho c^2 c^3 \alpha V_p \frac{I(BC) \sqrt{1 - B^2 c^2}}{E^*(BC)}$$

The lateral-force coefficient is formed by division by $\frac{1}{2} \rho V^2 S$, and the derivative with respect to $pb/2V$ is the stability derivative C_{Y_p} . It is

$$C_{Y_p} = \frac{2\pi\alpha}{3} \frac{I(BC) \sqrt{1 - B^2 c^2}}{E^*(BC)} \quad (18)$$

The yawing moment of the leading-edge suction about the vertex of the triangle is

$$N_0 = - \int_{x=0}^c (f_R - f_L) x \sqrt{1 + c^2} d(x \sqrt{1 + c^2})$$

$$= - \frac{\pi}{4} \rho c^2 c^4 \alpha V_p (1 + c^2) \frac{I(BC) \sqrt{1 - B^2 c^2}}{E^*(BC)}$$

The moment about the reference point $\left(\frac{2}{3}c, 0, 0\right)$ is

$$N = N_0 + \frac{2}{3} c Y$$

$$= - \frac{\pi}{36} \rho c^2 c^4 \alpha V_p (1 + 9c^2) \frac{I(BC) \sqrt{1 - B^2 c^2}}{E^*(BC)}$$

The yawing-moment coefficient is formed by division by $\frac{1}{2} \rho V^2 S b$, and the derivative with respect to $pb/2V$ is the stability derivative C_{n_p} . It is

$$C_{n_p} = - \pi\alpha \left(\frac{1}{9A} + \frac{A}{16} \right) \frac{I(BC) \sqrt{1 - B^2 c^2}}{E^*(BC)} \quad (19)$$

DERIVATIVES $C_{Y\beta}$, $C_{n\beta}$, C_{Yr} , and C_{nr}

According to the discussion on $C_{l\beta}$, a small angle of yaw or sideslip ($\beta \ll \frac{1}{M}$) does not alter the surface potential expressed relative to body axes. As a consequence the induced velocity distribution is unchanged. Thus the initially symmetric distribution of leading-edge suction persists in sideslip, and the derivatives $C_{Y\beta}$ and $C_{n\beta}$ are zero.

The surface potential relative to body axes is likewise unaltered in the first approximation by a small angular velocity of yaw. Accordingly, insofar as the pressure forces are concerned, the derivatives C_{Yr} and C_{nr} are zero. Subsonic experience, however, suggests an appreciable C_{nr} -derivative (damping in yaw) from profile drag. This damping derivative has been evaluated in reference 1 as

$$C_{nr} = -C_{D0} \left(\frac{1}{6} + \frac{4}{9A^2} \right) \quad (20)$$

RESULTS AND DISCUSSION

The formulas that have been obtained for the various stability derivatives are collected in table I. Derivatives obtained elsewhere are included for completeness, and the source is indicated in each instance. Expressions are given for two systems of coordinate axes. In the first column are shown the derivatives relative to the principal body axes of figure 2 with origin a distance $\frac{2}{3}c$ from the vertex of the triangle. In the second column are shown the results relative to stability axes with origin a distance x_{cg} ahead of the $\frac{2}{3}c$ point. The relationship between the two systems of axes is shown in figure 3. Equations for transforming from body axes to stability axes are given in reference 8; the shift in origin results in additional terms.

In the transformation of the present results from principal body axes to stability axes terms of order $A^2/16$ and the more important terms of order α^2 are retained (see footnote, table I), whereas in reference 1 such terms are dropped as a consequence of the narrow vertex-angle approximation.

These results for an arbitrary vertex angle may be compared with the asymptotic values for the case of vertex angle approaching zero given in reference 1. The present results for principal axes are found to differ from the asymptotic values (except for small terms in A^2 and α^2) only in the acquisition of certain factors which are functions of BC . Thus the asymptotic values for $C_{L\alpha}$, $C_{L\dot{\alpha}}$, C_{m_q} , C_{l_β} , and C_{l_r} are multiplied

by $E''(BC)$; C_{m_q} is multiplied by $G(BC)$; C_{L_q} is multiplied by $H(BC)$;

C_{l_p} is multiplied by $I(BC)$; and C_{n_p} and C_{Y_p} are multiplied

by $\frac{I(BC) \sqrt{1 - B^2 C^2}}{E'(BC)} \equiv J(BC)$. The parameter $BC = \frac{\tan \epsilon}{\tan \mu}$ is the ratio of

tangent of the semivertex angle of the triangle to the tangent of the Mach angle. BC approaches zero, therefore, as the vertex angle approaches zero. The several functions $E''(BC)$, . . . , $J(BC)$ all approach unity as BC approaches zero, and thus the derivatives obtained herein approach the asymptotic values of reference 1 as the vertex angle goes to zero.

The variation of these stability derivatives with Mach number is contained entirely in the factors $E''(BC)$, . . . , $J(BC)$. The five factors are plotted against $BC = \frac{\tan \epsilon}{\tan \mu}$, the ratio of the tangent of the semivertex angle to the tangent of the Mach angle, in figure 4.

The derivatives apply to a wing of triangular plan form and zero thickness. The calculations are based on the assumption of potential flow with small disturbances, except in the case of the derivative C_{n_r} , in which skin friction is considered. The predicted infinite negative pressure acting on an edge of zero thickness to yield a finite suction force is, of course, a mathematical idealization. (The local violation of the assumption of small disturbances is not serious.) Subsonic experience indicates that with a suitably rounded edge a considerable leading-edge suction force may be realized in practice, with the theoretical value an upper limit. On the other hand, a sharp leading edge is known to cause loss of the leading-edge suction. The requirements of extreme thinness and a rounded leading edge (that is, appreciable radius of curvature) are evidently in conflict. Thus the degree of applicability of the yawing-moment and lateral-force derivatives to actual triangular wings is uncertain. A further limitation on validity, already elaborated on in the section on C_{l_r} , exists also for the derivatives with respect to yawing velocity. The analysis neglects the spanwise variation in Mach number caused by the yawing (but not the spanwise variation in velocity). The result is an error in the magnitude of the yawing

derivatives that is expected to vary from zero for $BC \rightarrow 0$ to an important amount for $BC \rightarrow 1$.

The potential ϕ satisfies the linearized equation of motion for the steady state but not the more general linearized equation for unsteady motion. This circumstance implies that the present expressions for the stability derivatives are suitable only for steady motions, motions with small accelerations, or sinuous motions of low frequency. This limitation is accepted in all stability work and may become serious only in cases of high-frequency oscillations such as flutter.

Langley Memorial Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Field, Va., November 6, 1947

R E F E R E N C E S

1. Ribner, Herbert S.: The Stability Derivatives of Low-Aspect-Ratio Triangular Wings at Subsonic and Supersonic Speeds. NACA TN No. 1423, 1947.
2. Stewart, H. J.: The Lift of a Delta Wing at Supersonic Speeds. Quarterly Appl. Math., vol. IV, no. 3, Oct. 1946, pp. 246-254.
3. Brown, Clinton E.: Theoretical Lift and Drag of Thin Triangular Wings at Supersonic Speeds. NACA TN No. 1183, 1946.
4. Gurevich, M. I.: Lift Force of an Arrow-Shaped Wing. Appl. Math. and Mech. (Moscow), vol. X, no. 4, 1946, pp. 513-520.
5. Brown, Clinton E., and Adams, Mac C.: Damping in Pitch and Roll of Triangular Wings at Supersonic Speeds. NACA TN No. 1566, 1948.
6. Heaslet, Max. A., Lomax, Harvard, and Jones, Arthur L.: Volterra's Solution of the Wave Equation as Applied to Three-Dimensional Supersonic Airfoil Problems. NACA TN No. 1412, 1947.
7. Jones, Robert T.: Properties of Low-Aspect-Ratio Pointed Wings at Speeds below and above the Speed of Sound. NACA TN No. 1032, 1946.
8. Glauert, H.: A Non-Dimensional Form of the Stability Equations of an Aeroplane. R. & M. No. 1093, British A.R.C., 1927, p. 10.

TABLE I.- STABILITY DERIVATIVES OF THIN TRIANGULAR WINGS AT SUPERSONIC SPEEDS

Derivative	Source (if not derived herein)	Principal body axes (origin at $\frac{2}{3}c$)	Stability axes ¹ (origin at distance x_{cg} ahead of $\frac{2}{3}c$ point)
C_{L_α}	References 2 to 4	$\frac{\pi A}{2} E''(BC)$	$\frac{\pi A}{2} E''(BC)$
$C_{L_{\dot{\alpha}}}$		$\frac{\pi A}{2} E''(BC)$	$\frac{\pi A}{2} E''(BC)$
C_{L_q}	Reference 5	$\frac{\pi A}{2} H(BC)$	$\frac{\pi A}{2} H(BC) + \pi A \frac{x_{cg}}{c} E''(BC)$
C_{m_α}	References 2 to 4	0	$-\frac{\pi A}{2} \frac{x_{cg}}{c} E''(BC)$
$C_{m_{\dot{\alpha}}}$		$-\frac{\pi A}{16} E''(BC)$	$-\frac{\pi A}{16} \left(1 + 8 \frac{x_{cg}}{c}\right) E''(BC)$
C_{m_q}	Reference 5	$-\frac{3\pi A}{16} G(BC)$	$-\frac{3\pi A}{16} G(BC) - \frac{\pi A}{2} \frac{x_{cg}}{c} H(BC) - \pi A \frac{x_{cg}^2}{c^2} E''(BC)$
C_{l_β}	Reference 6	$-\frac{\pi A}{3} E''(BC)$	$-\frac{\pi A}{3} E''(BC)$
C_{l_p}	Reference 5	$-\frac{\pi A}{32} I(BC)$	$-\frac{\pi A}{32} I(BC) + \frac{\pi A^2}{9A} \left(1 + 8 \frac{x_{cg}}{c}\right) [E''(BC) - J(BC)]$
C_{l_r}		$\pi A \left(\frac{1}{9A} + \frac{A}{16}\right) E''(BC)$	$\pi A \left[\left(\frac{1}{9A} + \frac{A}{16} + \frac{8}{9A} \frac{x_{cg}}{c}\right) E''(BC) + \frac{A}{32} I(BC)\right] - \alpha C_{D_0} \left(\frac{1}{6} + \frac{4}{9A^2}\right)$
C_{n_β}		0	$\frac{\pi A^2}{3} E''(BC)$
C_{n_p}		$-\pi A \left(\frac{1}{9A} + \frac{A}{16}\right) J(BC)$	$-\pi A \left[\left(\frac{1}{9A} + \frac{A}{16} + \frac{8}{9A} \frac{x_{cg}}{c}\right) J(BC) - \frac{A}{32} I(BC)\right] - \alpha C_{D_0} \left(\frac{1}{6} + \frac{4}{9A^2}\right)$
C_{n_r}	Reference 1	$-C_{D_0} \left(\frac{1}{6} + \frac{4}{9A^2}\right)$	$-C_{D_0} \left(\frac{1}{6} + \frac{4}{9A^2}\right) - \pi A^2 \left(\frac{1}{9A} + \frac{A}{16} + \frac{8}{9A} \frac{x_{cg}}{c}\right) [E''(BC) - J(BC)] - \pi A^2 \frac{A}{32} I(BC)$
C_{Y_β}		0	0
C_{Y_p}		$\frac{2\pi A}{3} J(BC)$	$\frac{2\pi A}{3} J(BC)$
C_{Y_r}		0	$-\frac{2\pi A^2}{3} J(BC)$

¹In the transformation from body axes terms of order α^2 have been neglected in comparison with unity, but terms of order α^2/A have been retained, since they may be appreciable for small values of A .

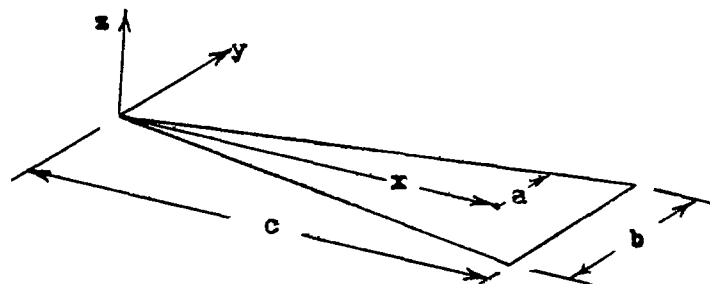


Figure 1.- Axes and notation used in analysis.

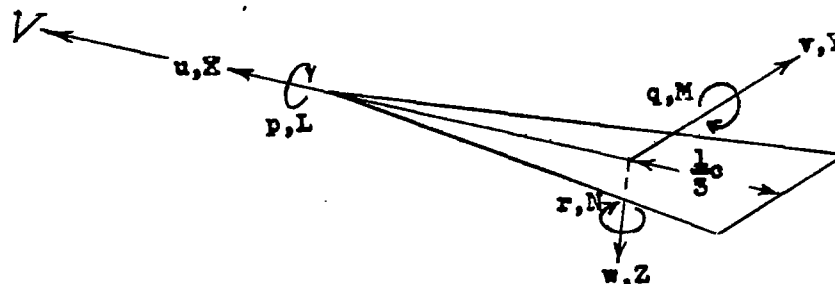


Figure 2.- Velocities, forces, and moments relative to principal axes with origin at $\frac{2}{3}c$.

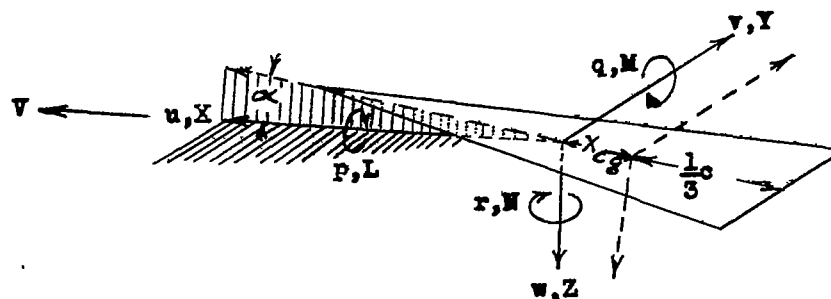


Figure 3.- Velocities, forces, and moments relative to stability axes with origin at $\frac{2}{3}c - x_{cg}$. Principal axes of figure 2 dotted in for comparison.

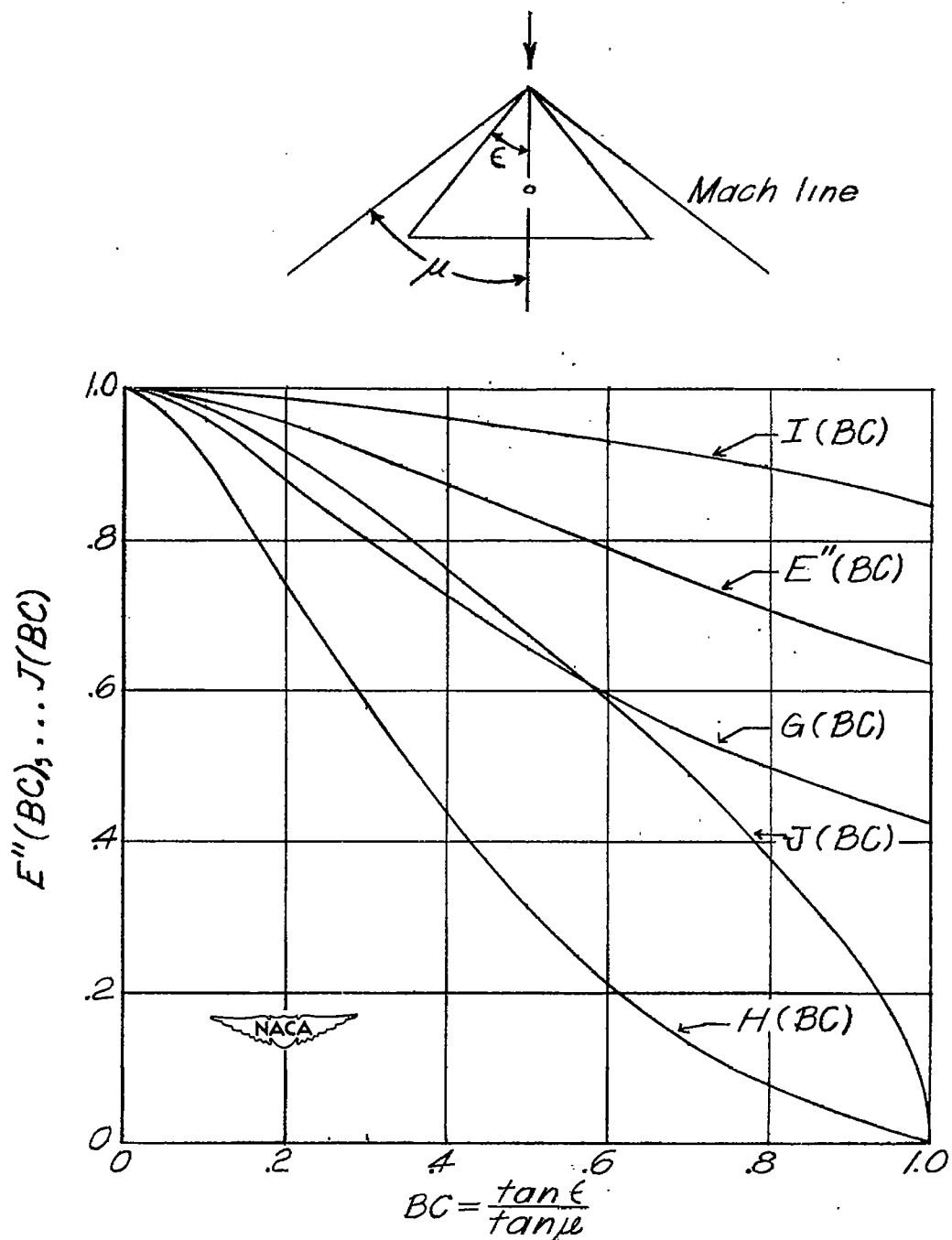


Figure 4. — Elliptic integral factors of the stability derivatives that determine their variation with Mach number.
(See Table I.)